

The Origin of Structures in Generalized Gravity

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Abstract

In a class of *generalized gravity theories* with general couplings between the scalar field and the scalar curvature in the Lagrangian, we can describe the *quantum generation* and the *classical evolution* of both the scalar and tensor structures in a simple and unified manner. An accelerated expansion phase based on the generalized gravity in the early universe drives microscopic quantum fluctuations inside a causal domain to expand into macroscopic ripples in the spacetime metric on scales larger than the local horizon. Following their generation from quantum fluctuations, the ripples in the metric spend a long period outside the causal domain. During this phase their evolution is characterized by their *conserved* amplitudes. The evolution of these fluctuations may lead to the observed large scale structures of the universe and anisotropies in the cosmic microwave background radiation.

1. Historical perspective

The classical evolution of structures in an expanding universe was first analyzed in the context of General Relativity in a classic study by E. M. Lifshitz in 1946 [1]. The theoretical introduction of an accelerated expansion (inflation) phase in the early universe [2] enables us to draw a coherent picture of the origin of the large scale structures in the universe: The ever-present microscopic vacuum quantum fluctuations become macroscopic during this acceleration phase and can subsequently develop into the observed large-scale structures.

Soon after the introduction of the field equation of General Relativity by A. Einstein in 1915 [3] and its immediate action formulation by D. Hilbert [4], H. Weyl, W. Pauli, and A. S. Eddington [5] considered modifications of the theory involving the addition of general curvature combinations to the action. Mach's principle motivated the Brans-Dicke gravity [6], and the notion of spontaneous symmetry breaking led to the idea of induced gravity [7]. Generalized forms of Einstein gravity almost always appear in any reasonable attempt to understand the quantum aspects of the gravity theory, and also naturally appear in the low energy limits of diverse attempts to unify gravity with other fundamental forces [8]. Consequently, it seems increasingly likely that the early stages of the evolution of the universe were governed by a gravity law more general than Einstein's theory.

2. Three stages

Current theoretical attempts to explain the origin of large scale structure in the universe can be described in three stages. We consider an expanding universe model modified by an accelerated expansion era in the early stage. The first stage is the *quantum generation stage*: structural seeds are generated from the ever present microscopic quantum fluctuations of the fields and the metric during the acceleration era. Due to the acceleration of cosmological expansion, quantum fluctuations residing in a causal domain are pushed outside the local horizon defined by a light travel distance. The second is the *classical evolution stage*: the magnified structures of quantum-mechanical origin become classical as their sizes become bigger than the local horizon. The evolution of fluctuations during this superhorizon stage is kinematic and is described by linear fluctuation theory based on classical relativistic gravity. In the later parts of this stage, as the expansion decelerates, the horizon scale overtakes the scales of the observationally relevant fluctuations, and makes these structures visible to the observer. The third stage is the *nonlinear evolution stage*: the smaller scale scalar type structures become nonlinear. Due to the extremely low level anisotropy of the cosmic microwave background radiation [9], nonlinear evolution is expected to start only on scales well within the horizon scale of the matter dominated era governed by Einstein gravity. Thus, the nonlinear evolution stage is usually handled in the Newtonian context. In this paper, we will address the role played by a class of generalized gravity theories in the first two stages and the unified perspective we can derive.

3. Generalized gravity and the perturbed universe

We consider a class of generalized gravity theories with an action

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2} f(\phi, R) - \frac{1}{2} \omega(\phi) \phi^{;a} \phi_{,a} - V(\phi) + L_m \right], \quad (1)$$

where f is a general algebraic function of the scalar (dilaton) field ϕ and the scalar curvature R , and ω and V are general functions of ϕ ; L_m is an additional matter part of the Lagrangian. Equation (1) includes the following generalized gravity theories as subsets: (a) generally coupled scalar fields, (b) generalized scalar tensor theories which include the Brans-Dicke theory, (c) induced gravity, (d) the low energy effective action of string theory, (e) $f(R)$ gravity, etc. Einstein gravity is a case with $f = R$ and $\omega = 1$. Each gravity theory in (a-e) without L_m corresponds to a single component system; the gravity theory in (1) without L_m in general corresponds to a two field system.

As the background universe, we consider a spatially homogeneous and isotropic metric with a vanishing spatial curvature and cosmological constant. Since the structures in the first two stages are assumed to be linear, they can be handled by perturbations of the model universe. The metric of the perturbed universe, including *the most general* scalar, vector and tensor perturbations, can be written as

$$ds^2 = -(1 + 2\alpha) dt^2 - a^2 (\beta_{,\alpha} + B_\alpha) dt dx^\alpha + a^2 \left[\delta_{\alpha\beta} (1 + 2\varphi) + 2\gamma_{,\alpha|\beta} + 2C_{(\alpha|\beta)} + 2C_{\alpha\beta} \right] dx^\alpha dx^\beta. \quad (2)$$

A scalar structure is characterized by $\alpha(\mathbf{x}, t)$, $\beta(\mathbf{x}, t)$, $\varphi(\mathbf{x}, t)$, and $\gamma(\mathbf{x}, t)$; the transverse $B_\alpha(\mathbf{x}, t)$ and $C_\alpha(\mathbf{x}, t)$, and the transverse-trace-free $C_{\alpha\beta}(\mathbf{x}, t)$ describe vector and tensor structures, respectively (the perturbed order metric variables have 10 degrees of freedom). When we consider a perturbed spacetime since we are dealing with two metric systems (one is a perturbed metric and the other is a fictitious unperturbed metric) we have the freedom of gauge choices. Due to the homogeneity of the background model, without losing generality, we can choose the spatial gauge conditions $\gamma \equiv 0 \equiv C_\alpha$ (thus fixing three degrees of freedom) which completely fix the spatial gauge degrees of freedom. Under this spatial gauge condition all the remaining scalar perturbation variables are *spatially* gauge invariant, and the remaining vector perturbation variable is also gauge invariant; tensor perturbation variables are naturally gauge invariant. The remaining temporal gauge condition with one degree of freedom only affects the scalar perturbation. In the gauge ready method, this temporal gauge condition can be used as an advantage in handling the scalar type perturbation by choosing the gauge condition according to the mathematical convenience of each individual problem [10]. Except for the synchronous gauge condition which fixes $\alpha \equiv 0$ the rest of the fundamental temporal gauge condition completely fixes the temporal gauge mode, thus each variable in such gauge condition corresponds to a gauge invariant combination of the considered variable and the variable used in the gauge condition; see below (3).

After fixing the functional forms of $f(\phi, R)$, $\omega(\phi)$, $V(\phi)$, and the equation of state, the equations for the background will lead to a solution for the cosmic scale factor $a(t)$. Due to the high symmetry in the background, all three types of perturbations *evolve independently of each other*. A *vector perturbation* is trivially described by a conservation of the angular momentum: for a vanishing anisotropic stress we have $a^3(\mu + p) \cdot a \cdot v_\omega \sim$ constant in time, where $\mu(t)$, $p(t)$, and $v_\omega(\mathbf{x}, t)$ are the background energy density and pressure, and the vorticity part of the matter velocity in L_m . Remarkably, the generalized nature of the gravity does not affect this result which is valid even considering the Ricci-curvature square term in the action, see [11].

4. The classical evolution of scalar and tensor structures

For the scalar field we let $\phi(\mathbf{x}, t) = \phi(t) + \delta\phi(\mathbf{x}, t)$. When we consider a scalar perturbation the following gauge invariant combination plays an important role

$$\delta\phi_\varphi \equiv \delta\phi - \frac{\dot{\phi}}{H} \varphi \equiv -\frac{\dot{\phi}}{H} \varphi_{\delta\phi}. \quad (3)$$

$\delta\phi_\varphi$ is the same as $\delta\phi$ in the uniform-curvature gauge ($\varphi \equiv 0$), and $\varphi_{\delta\phi}$ is the same as φ in the uniform-field gauge ($\delta\phi \equiv 0$); for the gauge transformation property of each variable see [10]. The perturbed action to the second order in the perturbation variables can be arranged in a remarkably simple and unified form (for derivation, see [12, 13, 14])

$$\delta^2 S = \frac{1}{2} \int a^3 Q \left(\dot{\Phi}^2 - \frac{1}{a^2} \Phi^{|\gamma} \Phi_{,\gamma} \right) dt d^3x, \quad (4)$$

where for scalar and tensor perturbations, respectively, we have ($F \equiv \partial f / \partial R$)

$$\Phi = \varphi_{\delta\phi}, \quad Q = \frac{\omega\dot{\phi}^2 + \frac{3\dot{F}^2}{2F}}{\left(H + \frac{\dot{F}}{2F}\right)^2}; \quad \Phi = C_{\beta}^{\alpha}, \quad Q = F. \quad (5)$$

The non-Einstein nature of the theory is present in the parameter Q . The equation of motion becomes

$$\frac{1}{a^3 Q} (a^3 Q \dot{\Phi})' - \frac{1}{a^2} \nabla^2 \Phi = 0. \quad (6)$$

This has a general large scale solution

$$\Phi = C - D \int_0^t \frac{dt}{a^3 Q}, \quad (7)$$

where $C(\mathbf{x})$ and $D(\mathbf{x})$ are the integration constants for the growing and decaying modes, respectively. This solution is valid for general $V(\phi)$, $\omega(\phi)$, and $f(\phi, R)$, and expresses perturbation evolution in a remarkably simple unified form; for the scalar type perturbation these results are valid for a single component subclass of (1) in (a-e) without L_m , whereas for the tensor type perturbation they are valid for the general action in (1). It is noteworthy that the growing mode of Φ (thus, $\varphi_{\delta\phi}$ and $C_{\alpha\beta}$) is conserved in the large scale limit *independently* of the specific gravity theory under consideration. It follows that the classical evolutions of very large scale perturbations are characterized by conserved quantities. These are *conserved even under the changing gravity theories* as long as the gravity theories belong to (a-e) for the scalar perturbation, and for general theory in (1) for the gravitational wave. [This conserved behavior also applies for sufficiently large scale perturbations during the fluid eras in Einstein gravity models; in the fluid era the defining criteria for considering a perturbation to be large scale are the Jeans scale (sound horizon) for a scalar structure and the visual horizon for a gravitational wave, [10].] The integration constant $C(\mathbf{x})$ *encodes the information about the spatial structure* of the growing mode. Thus, in order to have information about large scale structure, we need information about $\Phi = C(\mathbf{x})$ which must have been generated in some early evolutionary stage of the universe.

5. Quantum generations

In order to handle the quantum mechanical generations of scalar structures and gravitational waves, we regard the perturbed parts of the metric and matter variables as Hilbert space operators, $\hat{\Phi}$. Having the perturbed action in (4) the process of quantization and the derivation of quantum fluctuations are straightforward. The correct normalization of the equal time commutation relation follows from (4) as

$$[\hat{\Phi}(\mathbf{x}, t), \dot{\hat{\Phi}}(\mathbf{x}', t)] = \frac{i}{a^3 Q} \delta^3(\mathbf{x} - \mathbf{x}'). \quad (8)$$

[In the quantization of the gravitational wave we need to take into account of the two polarization states properly; we ignore this minor complication, see [15, 13].]

For $a\sqrt{Q} \propto \eta^q$ ($d\eta \equiv dt/a$) we have an exact solution for the mode function

$$\Phi_{\mathbf{k}}(\eta) = \frac{\sqrt{\pi|\eta|}}{2a\sqrt{Q}} \left[c_1(\mathbf{k}) H_{\nu}^{(1)}(k|\eta|) + c_2(\mathbf{k}) H_{\nu}^{(2)}(k|\eta|) \right], \quad \nu \equiv \frac{1}{2} - q, \quad (9)$$

where according to (8) we have $|c_2(\mathbf{k})|^2 - |c_1(\mathbf{k})|^2 = 1$; the freedom in c_1 and c_2 indicates the dependence on the vacuum state. The power spectrum based on the vacuum expectation value is

$$\mathcal{P}_{\hat{\Phi}}^{1/2}(\mathbf{k}, \eta) = \sqrt{\frac{k^3}{2\pi^2}} |\Phi_{\mathbf{k}}|. \quad (10)$$

In the large scale limit we have, for $\nu \neq 0$ and $\nu = 0$, respectively:

$$\mathcal{P}_{\hat{\Phi}}^{1/2}(\mathbf{k}, \eta) = \frac{H}{2\pi} \frac{\Gamma(\nu)}{\Gamma(3/2)} \frac{1}{aH|\eta|} \left(\frac{k|\eta|}{2} \right)^{3/2-\nu} |c_2(\mathbf{k}) - c_1(\mathbf{k})| \frac{1}{\sqrt{Q}}, \quad (11)$$

$$\mathcal{P}_{\hat{\Phi}}^{1/2}(\mathbf{k}, \eta) = \frac{2\sqrt{|\eta|}}{a} \left(\frac{k}{2\pi} \right)^{3/2} \ln(k|\eta|) \times |c_2(\mathbf{k}) - c_1(\mathbf{k})| \frac{1}{\sqrt{Q}}. \quad (12)$$

[For the gravitational wave, in order to get the correct numerical factor we need to take into account of the two polarization states properly, see [13].] In (8-12) the non-Einstein nature of the theory only appears in the parameter Q . Although the simplest vacuum state with $c_2 = 1$ and $c_1 = 0$ is often preferred in the literature, the power spectrums in (11,12) express the possible dependence on the general vacuum state.

The condition used to get (9) may look special. However, as a matter of fact, it includes most of the prototype inflation models investigated in the literature. The exponential ($a \propto e^{Ht}$) and the power-law ($a \propto t^p$) expansions realized in Einstein gravity with a minimally coupled scalar field lead to $\nu = \frac{3}{2}$ and $\nu = \frac{1-3p}{2(1-p)}$, respectively [16]. The pole-like inflations ($a \propto |t_0 - t|^{-s}$) realized in the generalized gravities in (b)-(d) with the vanishing potential lead to $\nu = 0$; these include the pre-big bang scenario based on the low energy effective action of the string theory [17].

As a perturbation scale reaches superhorizon size and the perturbation evolution enters the classical regime we can match the power spectrum in (11,12) with the classical one based on the spatial averaging. The later large scale perturbation evolution is characterized by the conserved behavior of Φ . Information about the classical structures can be recovered from Φ at the second horizon crossing epoch during the ordinary matter dominated era. At this point it provides the initial conditions for the nonlinear evolution stage. This completes the connection between quantum fluctuations in the early universe and large scale structures during the current epoch. The exponential expansion and the large p limit of the power-law expansion lead to scale invariant spectra for density perturbations and gravitational waves which conform with observations, whereas the pole-like inflation models based on the various generalized gravity with a vanishing potential lead to high power on small scales, see [13, 18].

6. Discussions

In this paper we have presented a unified way of describing the quantum generation and the classical evolution of scalar structures and gravitational waves in a class of generalized gravity theories. A rigorous treatment is made possible by two main ingredients: first, the general conservation behavior on large scales (7) which makes the classical evolution simple, and second, the exact solution in some generic expansion stages in (9) which makes the quantum perturbation generation simple. One underlying reason for these simple results in generalized gravity is the conformal equivalence between (1) and Einstein gravity with a minimally coupled scalar field, ignoring L_m [19, 12]. For nonvanishing L_m two theories mathematically related by the conformal transformation would be physically different [20].

The gravitational action in (1) does not include a Ricci-curvature square term which naturally appears from one-loop quantum corrections, [8]. Gravity with a Ricci-curvature square term does not have the conformal transformation symmetry with Einstein gravity and may lead to different results for structure generation. In particular, the recently popular string theory offers the possibility that the observationally relevant structures leave the horizon near the end of a pole-like inflation (pre-big bang) stage where the higher order string quantum correction terms are important. Investigating structure generation processes in the strong quantum regime is an interesting *open problem* which is left for the future endeavor.

Acknowledgments

This work was supported in part by the KOSEF Grant No. 95-0702-04-01-3 and through the SRC program of SNU-CTP.

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